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B.Sc. Part 2 (Hons) 4th Paper

Exact equations (continued)

$\boxed{\text{RULE}}$ If $M\,dx + N\,dy = 0$ is homogeneous and

$$Mx + Ny \neq 0$$

then $$\text{I.F.} = \frac{1}{Mx+Ny}$$

Q Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$ — (1)

Soln → Here $M = x^2y$ and $N = x^3+y^3$

The given equation is homogeneous.

Also $Mx + Ny = x \cdot x^2y - y(x^3+y^3)$

$$= x^3y - x^3y - y^4 = -y^4 \neq 0$$

$\therefore$ $$\text{I.F.} = \frac{1}{Mx+Ny} = \frac{-1}{y^4}.$$

Multiplying the given equation with I.F., we get

$$\Rightarrow \frac{x^2y}{-y^4}\,dx + \frac{x^3+y^3}{y^4}\,dy = 0$$

$$\Rightarrow -\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \quad \text{--- (2)}$$

Now, this equation has become exact,

as $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x^2}{y^3}\right)$

$$= -x^2 \cdot \frac{(-3)}{y^4} = \frac{3x^2}{y^4}$$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3}{y^4} + \frac{1}{y}\right) = \frac{3x^2}{y^4}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ eq (2) is exact

So, solution of eq (2) is

$\int [M\,dx]$ (treating y as constant)

$+ \int$ [those terms of N free from x] $dy = 0$

$$\Rightarrow -\frac{1}{y^3}\int x^2\,dx + \int \frac{1}{y}\,dy = 0$$

$$\Rightarrow -\frac{1}{y^3}\cdot\frac{x^3}{3} + \log y = K$$

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$\boxed{\text{RULE}}$

If the differential equation is of the form

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0$$

where $f_1(xy) \neq f_2(xy)$

then

$$I.F. = \frac{1}{Mx - Ny}$$

~~Solution Q:-~~

Q. Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$.

Soln: The given equation is of the form

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0$$

$\therefore\ Mx = xy[xy + 2x^2y^2)$

$Ny = xy(xy - x^2y^2)$

$\therefore\ Mx - Ny = 3x^3y^3$

$\therefore\ I.F = \dfrac{1}{Mx - Ny} = \dfrac{1}{3x^3y^3}$

Multiplying the given equation with I.F, we get

$$\Rightarrow \frac{y(xy + 2x^2y^2)}{3x^3y^3}\,dx + \frac{x(xy - x^2y^2)}{3x^3y^3}\,dy = 0$$

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$$\Rightarrow \frac{1+2xy}{3x^2y}\,dx + \frac{1-xy}{3xy^2}\,dy = 0 \quad \text{--- (2)}$$

It (2) is exact as

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1+2xy}{3x^2y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)$$

$$= \frac{-1}{3x^2y^2}$$

and $$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1-xy}{3xy^2}\right) = \frac{\partial}{\partial x}\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)$$

$$= \frac{-1}{3x^2y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Soln of eq (2) is given by

$$\int [M\,dx] \text{ treating } y \text{ as constant} + \int [\text{terms free from } x]\,dy = 0$$

$$\Rightarrow \int \left(\frac{1+2xy}{3x^2y}\right) dx + \int \frac{-1}{3y}\,dy = 0$$

$$\Rightarrow \frac{1}{3y}\int \frac{dx}{x^2} + \frac{2}{3}\int \frac{dx}{x} - \frac{1}{3}\int \frac{dy}{y} = 0$$

$$\Rightarrow -\frac{1}{xy} + 2\log x - \log y = K$$